Application of Householder Transformation to Matrix Simplification

Of course, despite all of the theoretical advantages of orthogonal matrices, Householder (reflection) transformations would be of only limited interest unless multiplying by them also produced some practically useful simplification. The latter is in fact the case, since it is fairly easy to construct Householder transformations with the properties that, for any given n-dimensional column vector $\mathbf{a}$ and for any $k < n$:

(i.) Multiplying $\mathbf{a}$ (on the left) by $\mathbf{Q}$ will not affect $a_1, \ldots, a_{k-1}$,

(ii.) Multiplying $\mathbf{a}$ by $\mathbf{Q}$ will change $a_{k+1}, \ldots, a_n$ to all zeros, and

(iii.) Multiplying any other vector $\mathbf{b}$ by $\mathbf{Q}$ will also leave $b_1, \ldots, b_{k-1}$ unchanged.

Building such a matrix is actually quite straightforward, since (i.) and (ii.) immediately imply:

$$\mathbf{Q}\mathbf{a} = \left(\mathbf{I} - 2\,\frac{\mathbf{u}\mathbf{u}^T}{\mathbf{u}^T\mathbf{u}} \right) \mathbf{a} = \mathbf{a} - 2\,\frac{\mathbf{u}^T\mathbf{a}}{\mathbf{u}^T\mathbf{u}}\mathbf{u}$$

$$= \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{k-1} \\ a_k \\ a_{k+1} \\ \vdots \\ a_n \end{bmatrix} - 2\,\frac{\mathbf{u}^T\mathbf{a}}{\mathbf{u}^T\mathbf{u}} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{k-1} \\ u_k \\ u_{k+1} \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{k-1} \\ \tilde{a}_k \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

and therefore (i.) and (ii.) can be immediately ensured if:

$$u_i = 0 \;\;,\;\; i = 1, \ldots, (k-1) \;\;, \qquad 2\,\frac{\mathbf{u}^T\mathbf{a}}{\mathbf{u}^T\mathbf{u}} \;=\; 1 \quad \text{and} \quad u_i = a_i \;\;,\;\; i = (k+1), \ldots, n$$

A small amount of algebra applied to the middle equation here then yields:

$$u_k = a_k \pm \sqrt{\sum_{i=k}^{n} a_i^2}$$

where, to minimize cancelation errors, the sign of the square root is chosen to be same as the sign of a_k. This result can be summarized as:

$$
\mathbf{u} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ a_k \pm \sqrt{\sum_{i=k}^{n} a_i^2} \\ a_{k+1} \\ \vdots \\ a_n \end{bmatrix}
$$

Moreover, note that then, for any other vector than $\mathbf{a}$,

$$
\mathbf{Q}\,\mathbf{b} \;=\; \mathbf{b} - 2\,\frac{\mathbf{u}^T\mathbf{b}}{\mathbf{u}^T\mathbf{u}}\mathbf{u} \;=\; \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{k-1} \\ b_k \\ b_{k+1} \\ \vdots \\ b_n \end{bmatrix} - 2\,\frac{\mathbf{u}^T\mathbf{b}}{\mathbf{u}^T\mathbf{u}} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ u_k \\ u_{k+1} \\ \vdots \\ u_n \end{bmatrix} \;=\; \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{k-1} \\ \tilde{b}_k \\ \tilde{b}_{k+1} \\ \vdots \\ \tilde{b}_n \end{bmatrix}
$$

where the tildes indicate changed elements. But this means multiplying any other vector (on the left) by $\mathbf{Q}$ will leave the first $(k-1)$ elements of that vector unchanged as well, i.e. Property (iii.) is also automatically satisfied! Moreover, if $\mathbf{A}$ is any matrix, then $\mathbf{Q}\,\mathbf{A}$ will have its first $(k-1)$ rows unchanged!

Even more interesting is the fact that for any other vector that already has zeros in positions k through n,

$$
\mathbf{u}^T\mathbf{z} \;=\; [\,0 \quad 0 \quad \cdots \quad 0 \quad u_k \quad u_{k+1} \quad \cdots \quad u_n\,] \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_{k-1} \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \;=\; 0
$$

and therefore

$$
\mathbf{Q}\mathbf{z} \;=\; \mathbf{z} - 2\,\frac{\mathbf{u}^T\mathbf{z}}{\mathbf{u}^T\mathbf{u}}\mathbf{u} \;=\; \mathbf{z} \;-\; 0\,\mathbf{u} \;\equiv\; \mathbf{z}
$$

i.e. the vector will be totally unchanged by the multiplication!

Given all of these results, we can then apply sequence of such transformations to a general matrix to, in order, zero out first the elements below a certain position in the first column, then those below the succeeding position in the second, without "messing up" the already zeroed out elements, etc., and so reduce the matrix to any one of several simpler forms.